MAE106 Mechanical Systems Laboratory
Controller Design Assignment

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Part 1: System Analysis

a) Write down the closed-loop transfer function relating the output θ to the input θ_d.

Starting with $\ddot{\theta} = \frac{g(M+m)}{Ml}\theta + \frac{1}{Ml}u$, substitute in PD control law $u = k_p(\theta_d - \theta) - k_d\dot{\theta}$ to get $\Rightarrow Ml\ddot{\theta} + k_d\dot{\theta} + (k_p - g(M + m))\theta = k_p\theta_d$. Next taking the laplace transform assuming zero initial conditions gives:

$$\frac{\Theta(s)}{\Theta_d(s)} = \frac{k_p}{Mls^2+k_ds+(k_p-g(M+m))}$$

b) Write down the poles of the closed-loop system as a function of the control gains and model parameters.

The poles are found by using the quadratic equation on $Mls^2 + k_ds + (k_p - g(M + m))$:

$$s_{1,2} = -\frac{k_d}{2Ml} \pm \frac{1}{2Ml}\sqrt{k_d^2 - 4Ml(k_p - g(M + m))}$$

c) What is the condition on k_p such that the poles are complex (i.e., for what values of k_p will the poles have non-zero imaginary parts)?

For the poles to have non-zero imaginary points, the discriminant must be less than zero, so solving $k_d^2 - 4Ml(k_p - g(M + m)) < 0$ for k_p yields:

$$k_p > \frac{k_d^2}{4Ml} + g(M + m)$$

d) Write down the parameter values you chose.
   i) Cart Mass (M): 1.0 kg
   ii) Pendulum Mass (m): 0.5 kg
   iii) Pendulum Length (l): 1.0 m
   iv) Gravity (g): 9.81 m/s^2

Part 2: Controller Design

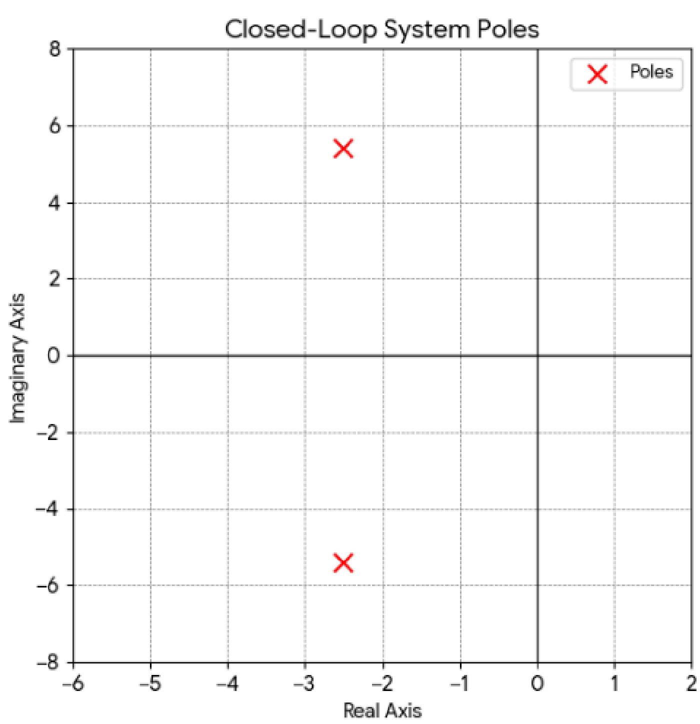
a) Write down the final kp and kd values you chose that stabilize the system.
   i) Kp = 50.0
   ii) Kd = 5.0

b) Given these gain values and the parameter values you chose, calculate the values of the poles of your closed-loop system and plot them on the complex plane. Drawing the figure by hand is fine.

$$s_{1,2} = -\frac{k_d}{2Ml} \pm \frac{1}{2Ml}\sqrt{k_d^{\,2} - 4Ml(k_p - g(M + m))}$$

$$s_{1,2} = -\frac{5}{2(1)(1)} \pm \frac{1}{2(1)(1)}\sqrt{5^2 - 4(1)(1)(50 - 9.81(1 + 0.5))}$$

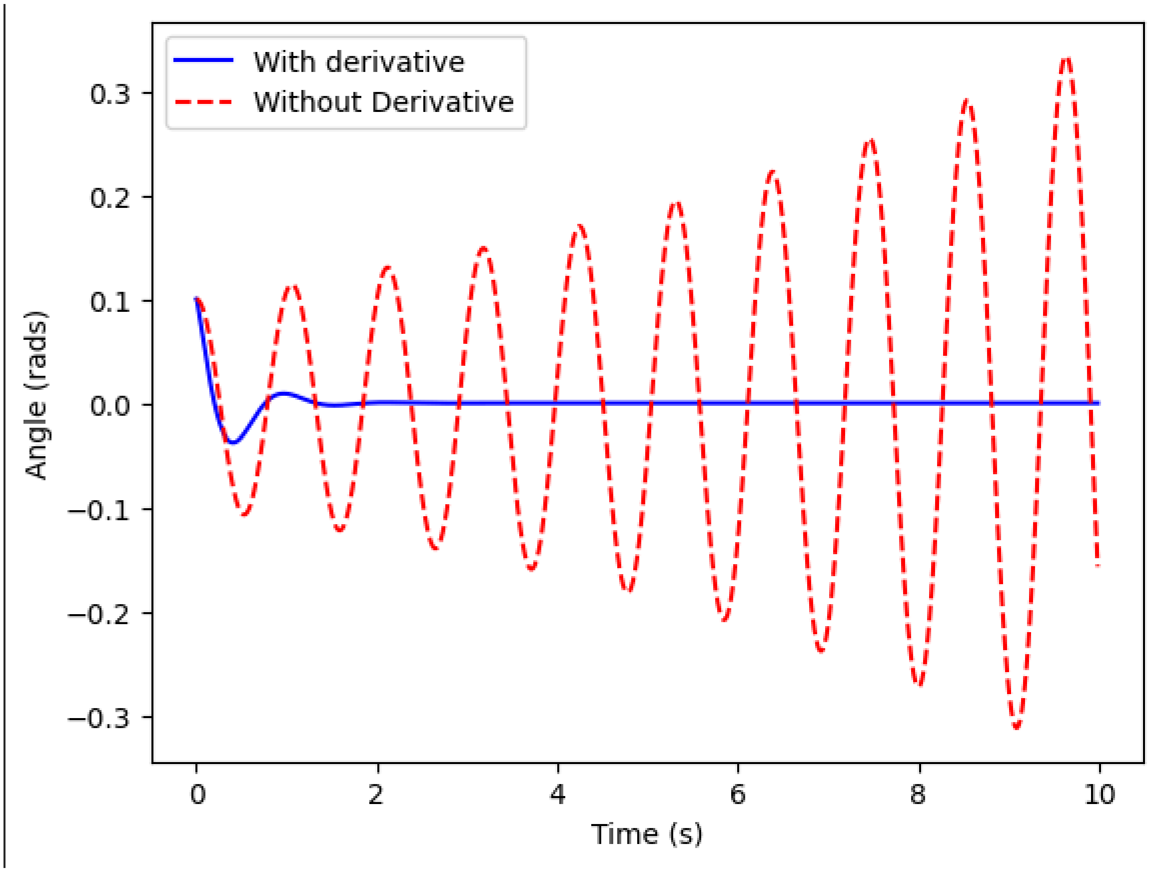
$$s_{1,2} = -2.5 \pm 5.388j$$



c) Is this system stable, marginally stable, or unstable? Is the system underdamped, critically damped, or overdamped?

The system is stable because the real part of the poles is strictly negative (-2.5). The system is underdamped because the poles are complex conjugates.

d) Include the figure generated above that shows the comparison between responses with a P controller versus a PD controller.



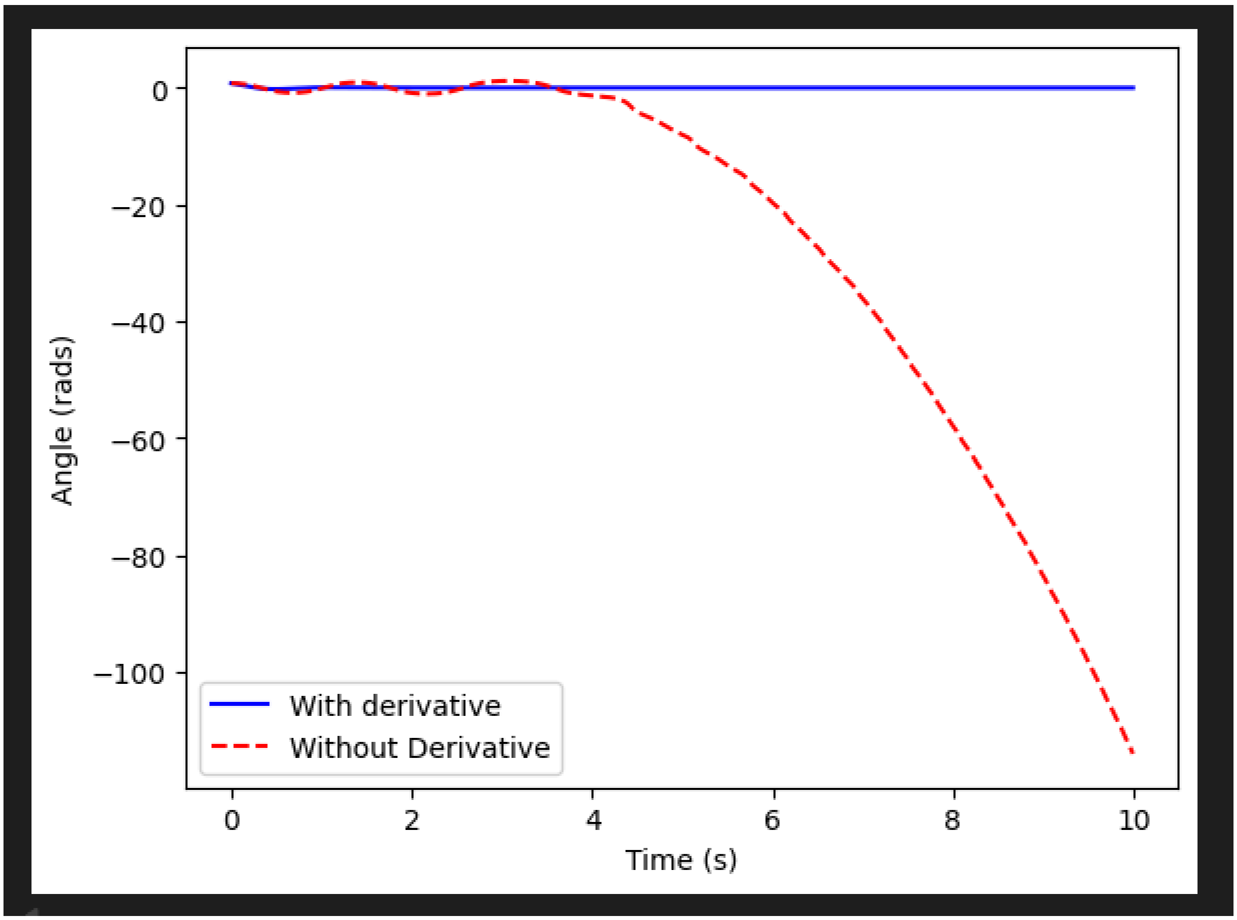
e) Would adding an integral term to the control law (i.e., make it a PID controller rather than a PD controller) improve the performance of this closed-loop system? Describe why or why not. Feel free to experiment with k_i values in the code.

Adding an integral term (Ki) is not necessary to stabilize this ideal inverted pendulum and would likely degrade transient performance. An integral term is designed to eliminate steady-state error from constant external disturbances. Because the goal is simply to maintain theta = 0 against gravity from an initial offset, introducing an integrator adds a pole at the origin, reducing the stability margin, increasing overshoot, and making the system more oscillatory.

f) Finally, change the initial condition for θ in Part 1 above. What is the largest initial angle for which the same PD controller you already tuned can stabilize the system? (You only need to update the initial condition in the code that defines 'pend' in Part 1, rerun that code, and rerun the code at the end of Part 2 that compares the P and PD controllers.)

Largest angle = 0.78 rad

g) Include the figure generated above comparing the P and PD controllers for this new initial condition.



Part 3: PID Controller Design Discussion

Based on your observations from Parts 1 & 2:

a) Describe the primary function of each of the three controller terms (P, I, and D) in shaping system performance.
   i) Proportional provides the primary restoring force to push the pendulum upright, producing a control effort proportional to the current error.
   ii) Integral accumulates past error to overcome constant, steady-state disturbances.
   iii) Derivative predicts future error based on the rate of change, providing necessary damping to reduce overshoot and settle oscillations.

b) Explain the trade-offs associated with increasing each controller gain.
   i) Increasing k_p: Speeds up the system response, but decreases the stability margin and increases oscillations.
   ii) Increasing k_i: Eliminates steady-state error entirely, but slows down the transient response and severely reduces system stability.
   iii) Increasing k_d: Increases damping and reduces overshoot, but a value too high will make the system sluggish and amplify high-frequency measurement noise.

c) Relate your observations to movement/placement of the system's poles and to the system's damping behavior.

   Increasing k_p pulls the complex poles further away from the real axis, increasing their imaginary component (natural frequency) and causing the high-frequency oscillations observed in the P-only simulation. Adding and increasing k_d pulls the poles deeper into the left-half plane (more negative real part), which increases the damping ratio, stops the continuous oscillations, and stabilizes the system.

d) Discuss whether all three terms (P, I, and D) are necessary for stabilizing the inverted pendulum.

No, only P and D are necessary. A proportional gain is strictly required to physically overcome the force of gravity ($k_p > g(M+m)$). Because the P-only system results in bounded but sustained/growing oscillations, the derivative gain is strictly required to add artificial damping. An integral gain is unnecessary for stabilization and actually degrades the stability margin.